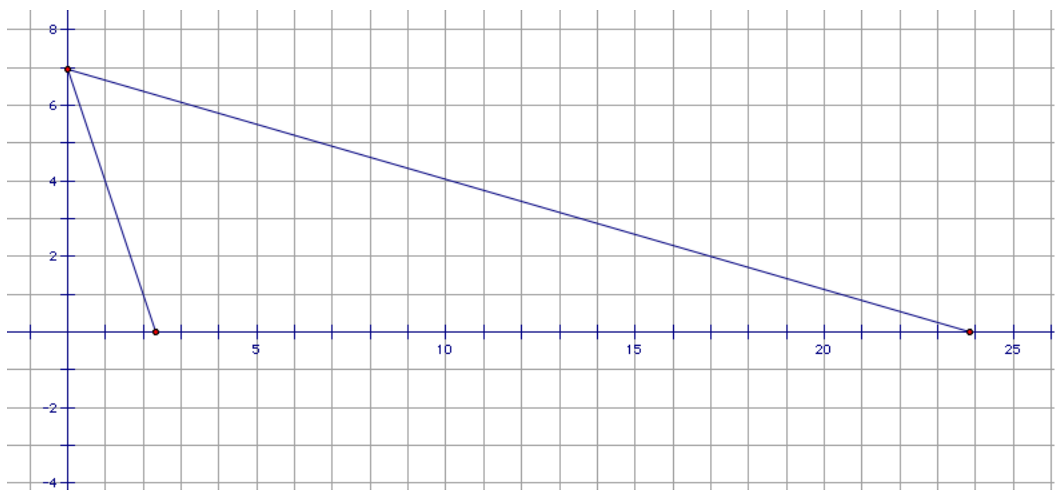
The University of Georgia
Department of Mathematics and Science Education
J. Wilson, EMAT 6680

EMAT 6680 - Assignment 4

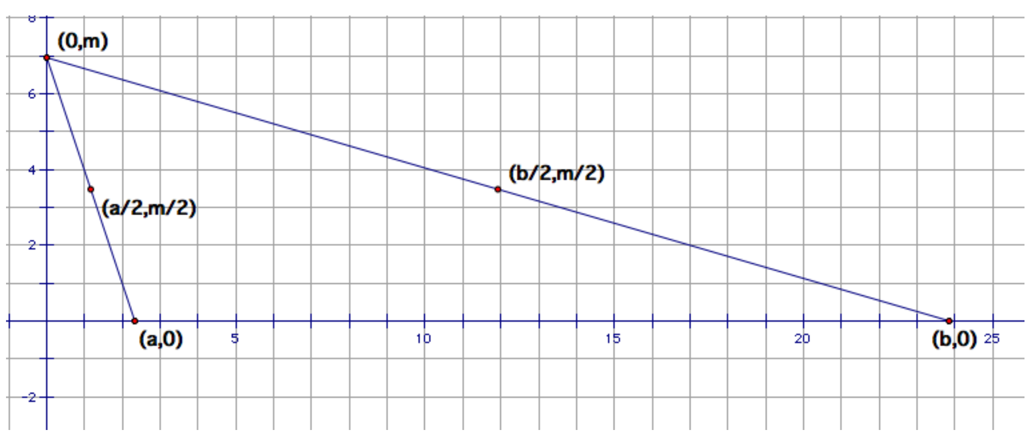
By Brandon Samples

Question: Prove that the three perpendicular bisectors of the sides of a triangle are concurrent.

Proof. Let T be any triangle and place the triangle with its base on the x-axis such that the third corner is on the y-axis. (See Figure)

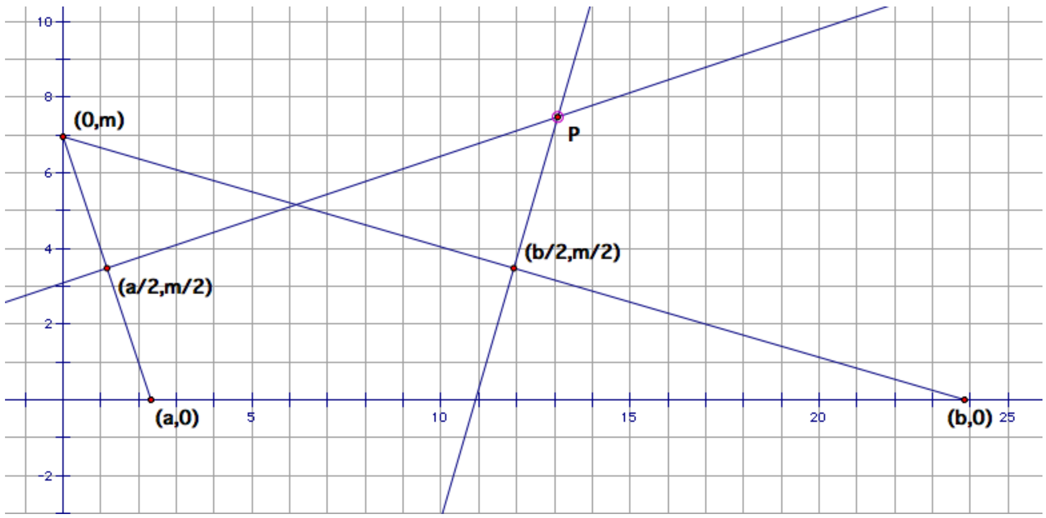


First, label the three corner points $(a, 0)$, $(b, 0)$ and $(0, m)$, and note that two of the midpoints are $\left(\frac{a}{2}, \frac{m}{2}\right)$ and $\left(\frac{b}{2}, \frac{m}{2}\right)$.



Now, the perpendicular bisectors going through these two midpoints are given by the equations:

$$y = \frac{a}{m}\left(x - \frac{a}{2}\right) + \frac{m}{2}$$
$$y = \frac{b}{m}\left(x - \frac{b}{2}\right) + \frac{m}{2}$$



Let P represent the intersection point. To determine the coordinates of P, let's set these equations equal to each other. Specifically, we have

$$\begin{aligned}
\frac{a}{m}\left(x - \frac{a}{2}\right) + \frac{m}{2} &= \frac{b}{m}\left(x - \frac{b}{2}\right) + \frac{m}{2} \\
\implies \frac{a}{m}\left(x - \frac{a}{2}\right) &= \frac{b}{m}\left(x - \frac{b}{2}\right) \\
\implies a\left(2x - a\right) &= b\left(2x - b\right) \\
\implies 2ax - a^2 &= 2bx - b^2 \\
\implies 2x(a-b) &= a^2 - b^2 \\
\implies 2x(a-b) &= (a+b)(a-b) \\
\implies 2x &= a + b \\
\implies x &= \frac{a+b}{2}.
\end{aligned}$$

Therefore, we see that the x-coordinate of P is $\frac{a+b}{2}$, hence the y-coordinate of P is

$$\frac{a}{m}\left(\frac{a+b}{2} - \frac{a}{2}\right) + \frac{m}{2} = \frac{a}{m}\left(\frac{b}{2}\right) + \frac{m}{2} = \frac{ab + m^2}{2m}.$$

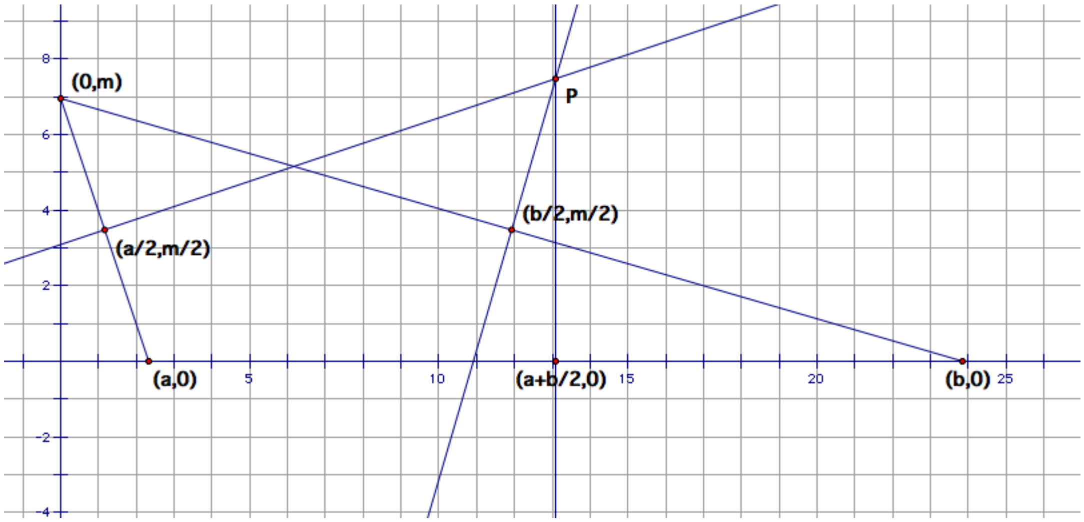
Now, the midpoint of the base has coordinates

$$\left(\frac{a+b}{2}, 0\right),$$

so the perpendicular bisector is the vertical line

$$x = \frac{a+b}{2}$$

and therefore passes through P.



□